Assignment 9

This homework is due *Tuesday* Nov 20.

There are total 28 points in this assignment. 25 points is considered 100%. If you go over 25 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 4.2, 5.1 in Bartle–Sherbert.

(1) Using arithmetic properties of limit, find the following limits.
  (a) [2pt] $\lim_{x\to 1} \frac{x^{100}+2}{x^{100}-2}$.
  (b) [2pt] $\lim_{x\to 0} \frac{(x+1)^{20}-1}{x}$.
  (c) [2pt] $\lim_{x\to c} \frac{(x-c+1)^2-1}{x-c}$.

(2) [3pt] (4.2.15) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. In addition, suppose $f(x) \geq 0$ for all $x \in A$, and let $\sqrt{f}$ be the function defined for $x \in A$ by $(\sqrt{f})(x) = \sqrt{f(x)}$. If $\lim_{x\to c} f$ exists, prove that $\lim_{x\to c} \sqrt{f} = \sqrt{\lim_{x\to c} f}$. (*Hint:* $a^2 - b^2 = (a-b)(a+b)$.)

(3) (a) [3pt] (4.2.5) Let f, g be defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim_{x\to c} g = 0$. Prove that $\lim_{x\to c} fg = 0$.
  (b) [2pt] ($\sim$4.2.12b) Determine whether $\lim_{x\to 0} x\cos(1/x^2)$ exists in $\mathbb{R}$.

(4) [3pt] (4.2.12) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume that $\lim_{x\to 0} f(x) = L$ exists. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathbb{R}$. (*Hint:* First note that $f(2x) = f(x) + f(x)$ for $x \in \mathbb{R}$. Also note that $f(x) = f(x-c) + f(c)$ for all $x, c \in \mathbb{R}$.)

(5) [2pt] (5.1.7+) (Local separation from zero) Let $A \subseteq \mathbb{R}$, $c \in A$, $f : A \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that for any $\alpha \in \mathbb{R}$ such that $0 < \alpha < f(c)$, there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c) \cap A$, then $f(x) > \alpha$.

(6) [2pt] (5.1.8) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $S = \{x \in \mathbb{R} \mid f(x) = 0\}$ be the "zero set" of f. If (x_n) is in S and $x = \lim(x_n)$, show that $x \in S$.

(7) [3pt] (5.1.13) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$ for $x \in \mathbb{Q}$ and $g(x) = x + 3$ for $x \notin \mathbb{Q}$. Find all points at which g is continuous.

(8) (5.1.9) Let $A \subseteq B \subseteq \mathbb{R}$, let $f : B \to \mathbb{R}$ and let $g = f|_A$ be the restriction of g to A (that is, $g(x) = f(x)$ for $x \in A$).
  (a) [2pt] If f is continuous at $c \in A$, show that g is continuous at c.
  (b) [2pt] Show by example that if g is continuous at c, it need not follow that f is continuous at c.